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Lattice Gases and Exactly Solvable Models

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We detail the construction of a family of lattice gas automata based on a model of 't Hooft, proceeding by use of symmetry principles to define first the kinematics of the model and then the dynamics. A spurious conserved quantity appears; we use it to effect a radical transformation of the model into one whose spacetime configurations are equivalent to the two-dimensional states of an exactly solvable statistical mechanics model, the symmetric eight-vertex model with parameters restricted to a disorder variety. We comment on the implications of this identification for the original lattice gas.

KEY WORDS: Lattice gases; spacetime lattice; spurious conserved quantities; solvable models; Yang–Baxter equation; disorder solutions; thermodynamic equilibrium.

1. INTRODUCTION

Lattice gas automata bear much the same relationship to the study of nonlinear dynamics that statistical mechanics models do to the study of equilibrium thermodynamics. There are historical differences, of course, many stemming from the ubiquity of computers and the consequent temptation to simulation during this era in which lattice gas automata have come to prominence. Structurally, however, the analogy is sound: in each case we have a discrete model of a (macroscopically) continuous physical system and it is primarily as a model of this physical system that the discrete model is of interest. One has, for example, the Ising model for bulk magnetization[1] and the eponymous ice model[2] on the one hand, and the FHP model for fluid flow[3,4] and the various lattice gas automata for diffusion processes[5] on the other.

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While there are other fascinating aspects of statistical mechanics models and lattice gas automata—as purely mathematical problems, as realizations of issues in theoretical computer science, as testing grounds for massively parallel computation, and as models potentially realizable at the level of device physics—it is doubtful that either would have attracted/attract so much attention were the models not so physically successful. The Ising model, for example, predicts critical exponents extremely well[1] and lattice gas automata models for fluid flow have successfully simulated as diverse phenomena as nonlaminar flow past an obstacle in three dimensions[6] and two-phase flow through a porous medium.[7]

The successes of lattice gas automata are particularly remarkable given the incompleteness of current theoretical understanding. The dominant question, of course, is how to design a lattice gas automaton to model a given physical system/collection of partial differential equations—the inverse compiler problem in the language of theoretical computer science. There are heuristics for this problem, based largely on symmetry principles, but no general algorithm.[8] Part of the difficulty is that given a lattice gas automaton model, even the powerful machinery of kinetic theory—the Boltzmann transport equation, the Chapman–Enskog expansion, etc.[9]—is insufficient to *prove* that a macrodynamics governed by a particular system of partial differential equations arises. Various assumptions, such as the existence of a local thermodynamic equilibrium, and approximations, such as the truncation of the asymptotic series in the Chapman–Enskog expansion, are commonly made.[3,10] One would like to understand to what extent these assumptions and approximations are justified. Although in many cases they seem to be, there are also phenomena, such as extraneous/spurious conserved quantities[4] and the very finiteness of the phase space, which suggest otherwise. These observations, too, should be understood.

The goal of this paper is to explain an approach toward answering some of these questions in the case of a particular model. It is based on the fact that for some models the analogy between lattice gas automata and statistical mechanics models is actually an *equivalence*. That is, the spacetime configurations of an n-dimensional lattice gas automaton are the $(n+1)$-dimensional states of a statistical mechanics model.[11] Also, we recall that the statistical mechanics models which provide the most theoretical insight into equilibrium thermodynamics are those which are *exactly solvable*, i.e., those for which the partition function or the free energy can be evaluated in closed form in the thermodynamic limit.[12] With these considerations in mind, in Sections 2 and 4 we detail the construction of a family of lattice gas automata based on a model of 't Hooft for a system with discrete local coordinate invariance.[13] In Section 3 we

note the existence of spurious conservation laws in this model and are motivated by this observation to transform it into one which, in Section 5, we show is equivalent to a statistical mechanics model. Moreover, in Section 6 we find that this model is exactly solvable. Finally, in Section 7, we consider the consequences of this identification, indicating how they may answer some of the questions raised in the previous paragraph.

2. KINEMATICS

The approach we outlined in the introduction requires a lattice gas automaton which is equivalent to a statistical mechanics model. To construct such a model, we will impose a collection of constraints forcing this equivalence; to do so, we will need to view the model from a somewhat unusual spacetime perspective. For this reason, 't Hooft's model for two-dimensional spacetime with discrete local coordinate invariance is a natural inspiration. In an effort to clarify which features of the lattice gas automaton are the consequences of which constraints, we will proceed as systematically as possible: in this section we set the kinematics of the model; in the next we explore the effect of requiring that the dynamics be local; and in Section 4 we construct the possible dynamics consistent with the imposed constraints. The exposition in this section and Section 4 is strongly influenced by 't Hooft's paper.[13]

With the exception of some quite recent, fascinating results in three dimensions,[14] almost all solvable statistical mechanics models are two dimensional. The underlying spatial lattice of the dynamical model, then, should be one dimensional, isomorphic to either $\mathbf{Z}$ or $\mathbf{Z}_N$ (the integers or the integers mod N) when we choose to avoid imposing boundary conditions at a wall. Time is also discrete, the nodes of the spatial lattice being updated simultaneously at discrete intervals.

Guided by the experience of field theory, we shall require the model to have parity (P) and charge-conjugation/time-reversal (CT) symmetry. For a one-dimensional lattice gas, P implies the existence of both left- and right-moving particles. To impose CT, we choose to have a gas with two types of particles: A and its CT dual, $\bar{A}$. We could choose to use a particle which was self-dual, of course, but this choice will allow more structure in the model.

Thus, at its most fundamental level, the kinematics of this model comprises a one-dimensional lattice on which two types of particles, A and $\bar{A}$, propagate one step to the left or right in discrete time intervals. (Although we could choose $|\mathbf{v}| > 1$, it will be clear after the next section that doing so would make no significant difference; multiple velocities, of course, would be a nontrivial generalization.) We further impose one additional

constraint: an *exclusion principle* mandating that there be at most one left-moving particle and at most one right-moving particle on a node at any time. Empty nodes are allowed by this exclusion principle; the configuration space is clearly $(3^2)^\omega$ on $\mathbf{Z}$ and $(3^2)^N$ on $\mathbf{Z}_N$.

3. SPURIOUS CONSERVATION LAWS

In the previous section we imposed discreteness, one-dimensionality, symmetry under P and CT, as well as an exclusion principle, to fix the kinematics of the lattice gas automaton we are constructing. Without yet specifying the dynamics, requiring locality of interactions already has significant consequences. By local interactions we mean that two particles interact only when they coincide in both space and time. In practice this means that at each time step all the particles first move one step to either the left or the right and then interact with whatever other particle may be occupying the same node of the lattice. At the next time step the particles resulting from these interactions move in the appropriate direction and the evolution continues.

Let us consider, then, the configuration of particles shown in Fig. 1. The right-moving particle labeled 1 interacts with the left-moving particle labeled 2 at time $t+1$, since they occupy the same node of the lattice; call the resulting particles $1'$ and $2'$. At the next time step the right-moving particle $1'$ and the left-moving particle 3 have bypassed each other, without interacting. The difference is, of course, that particles 1 and 2 started an even number of nodes apart and particle 3 an odd number of nodes from both. Since the separation (mod 2) of any two particles is preserved by the evolution, only those an even number of nodes apart can ever interact. Thus, if the underlying lattice is $\mathbf{Z}$, the system really consists of two uncoupled gases, one lying on the even nodes, the other on the odd. (Were $|\mathbf{v}| > 1$, some other, but similar, partition would obtain.) This is the origin

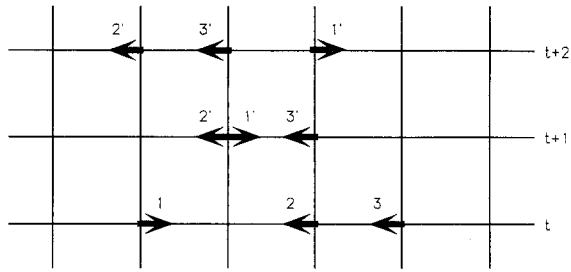

Fig. 1. A configuration of particles in the original formulation of the lattice gas automaton. Note that particle 1 interacts with particle 2 but not with particle 3.

of what have been called *spurious conservation laws* in other lattice gas automata[4]; it is clear that any function of the state of the even (odd) nodes which is conserved by the dynamics—particle number, say—is a conserved quantity which has no counterpart in the continuum limit.

The usual reaction to the discovery of such spurious conservation laws is to attempt to modify the model to break them. While this is often not difficult to do, it is rarely easy to show that *no* extraneous conserved quantities remain after all the obvious modifications. It is also rarely clear what effect the more subtle spurious conserved quantities have on statistical properties of the lattice gas automaton. The existence of any precludes ergodicity *a fortiori*, but experience with other finite configuration-space dynamical systems suggests that while this may be unavoidable, it does not necessarily prevent the development of desirable statistics such as a canonical ensemble measure.[15,16]

Rather than breaking the spurious symmetry we have found in this model, however, we choose to use it to effect a radical transformation of the model. Since the lattice gas automaton over $\mathbf{Z}$ splits into two uncoupled gases, as it does over $\mathbf{Z}_N$ when N is even, we can consider the uncoupled systems separately, as has been done for the Burgers' equation model of ref. 5 in ref. 17. That is, we consider a lattice gas automaton where the particles lie only on the even nodes, say. Rescaling the *spacetime* lattice by one-half and allowing the interactions to occur in the middle of the time steps, as shown in Fig. 2, we obtain a new model. A slightly different transformation applied to the lattice gas automaton over $\mathbf{Z}_N$ when N is odd produces the same new model: since by traversing the entire space a particle moves from an even node to an odd one, we simply take the double cover of the lattice to be the rescaled lattice. See Fig. 3.

Three typical spacetime diagrams of particle trajectories in the rescaled model are shown in Fig. 4. In each case they determine a tiling of spacetime by rectangles. The regularity of this spacetime configuration suggests a

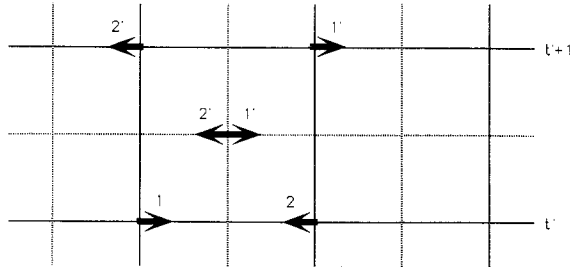

Fig. 2. Rescaling the lattice gas automaton with particles on the even nodes only.

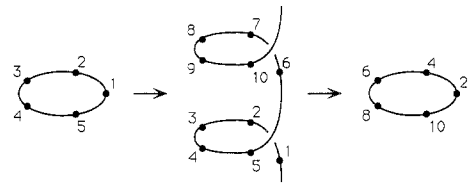

Fig. 3. For the lattice $\mathbf{Z}_N$ when N is odd, taking the double cover gives the lattice for the rescaled model.

further transformation of the model: Drawing a spacelike curve (one with tangent vector within $\pi/4$ of horizontal) which crosses the rectangles in an alternating sequence of left and right trajectories indicates that a simpler initial configuration (one with alternating left- and right-moving particles) would evolve identically in the first two cases shown. In the third case, because the lattice is periodic and there are different numbers of left- and right-moving particles, the spacelike curve is not Cauchy: it clearly cannot be used to determine an initial condition for the evolution. With the exception of this case, however, the original model is equivalent to the rescaled model with alternating initial conditions.

Now observe that the spacing between right (left)-moving particles is constant—the trajectories are parallel. Thus, if we store the spacings

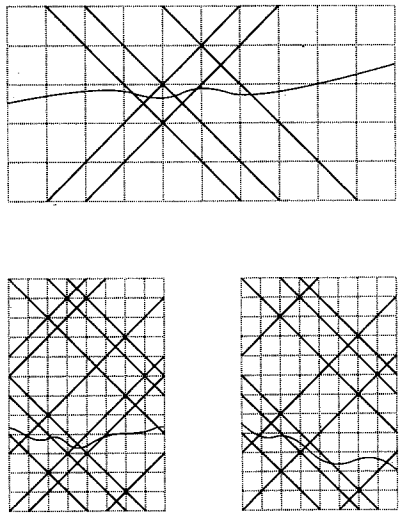

Fig. 4. Spacetime diagrams of particle trajectories in the rescaled models on $\mathbf{Z}$ and $\mathbf{Z}_N$. The spacelike curve in each of the diagrams crosses the trajectories in an alternating sequence of left- and right-moving particles. Since the second periodic diagram has different numbers of left- and right-moving particles, the spacelike curve is not Cauchy.

initially, they can be ignored throughout the evolution and restored at the end. But this means that we need only consider a lattice which is saturated, i.e., we can conformally rescale the spacetime diagram so that all the rectangles are squares of side $\sqrt{2}$ in rescaled lattice units and each node carries exactly one particle at each time step as shown in Fig. 5; there is now a complete exclusion principle: all the interactions occur between time steps and only one particle of any type lies on a given node at a particular time. This saturated, alternating, rescaled model is the one we shall be able to solve exactly. Inverting the transformations successively would reconstruct the original lattice gas automaton and reinterpret as properties of its evolution quantities derived from the solution.

4. DYNAMICS

Before solving this model we must complete its description; to this point we have not specified a dynamics, requiring only that it be local. Thus we must now determine the possible interaction rules for left- and right-moving A and $\bar{A}$ particles. These interactions must be P and CT invariant: CT implies that they be reversible—distinct configurations evolve to distinct configurations—and P implies that (a)symmetric configurations evolve to (a)symmetric configurations.

A deterministic interaction rule, therefore, necessarily consists of a pair of 1–1 maps

$$\{AA, \bar{A}\bar{A}\} \to \{AA, \bar{A}\bar{A}\}$$
$$\{A\bar{A}, \bar{A}A\} \to \{A\bar{A}, \bar{A}A\}$$

where the notation XY indicates a right (left)-moving X particle and a left (right)-moving Y particle before (after) the interaction. There are clearly four such rules, each of which satisfies P and CT invariance.

More generally, and also as suggested by the reduction from 't Hooft's original model, we can consider probabilistic interaction rules. These are most conveniently represented by a *scattering matrix*:

$$R := \begin{array}{c} \\ AA \\ A\bar{A} \\ \bar{A}A \\ \bar{A}\bar{A} \end{array} \begin{array}{c} \begin{array}{cccc} AA & A\bar{A} & \bar{A}A & \bar{A}\bar{A} \end{array} \\ \begin{pmatrix} p & 0 & 0 & 1-p \\ 0 & q & 1-q & 0 \\ 0 & 1-q & q & 0 \\ 1-p & 0 & 0 & p \end{pmatrix} \end{array}$$

which is really a fourth-order tensor with components R_{YZWX} representing the probability of the $WX \to YZ$ interaction. The components have the form indicated because of the constraints we have imposed:

$$R_{YZWX} \geqslant 0 \qquad \text{(positive probability)}$$

$$\sum_{YZ} R_{YZWX} = 1 \qquad \text{(particle number conservation)}$$

$$W = X,\ Y \neq Z \Rightarrow R_{YZWX} = 0 \qquad \text{(parity)}$$

$$R_{YZWX} = R_{ZYXW} \qquad \text{(parity)}$$

$$R_{YZWX} = R_{\bar{W}\bar{X}\bar{Y}\bar{Z}} \qquad \text{(charge conjugation/time reversal)}$$

Note that the probabilistic rules are a generalization of the deterministic ones: letting p and q take the values 0 and 1 generates the four deterministic rules. Also note that for $p \neq 1$, although total particle number is conserved, the interaction rule does not conserve the two types of particles separately.

5. EVOLUTION OF THE LATTICE GAS

The goal of analyzing any lattice gas is to understand its evolution, ultimately, to know its state at any time t given the initial state. For a probabilistic lattice gas one can, of course, expect no more than a distribution of final states. Considering the problem of computing this distribution will lead us directly to the equivalent statistical mechanics model.

In Section 3 we transformed the original lattice gas into one which is saturated, alternating, and rescaled. As this system evolves, each node carries left- and right-moving particles at alternate time steps, as we can see in Fig. 5. Thus, given the parity of the initial condition, there is only one bit of state information—A or $\bar{A}$—per node. Denoting this bit by $\sigma_n(t)$, on

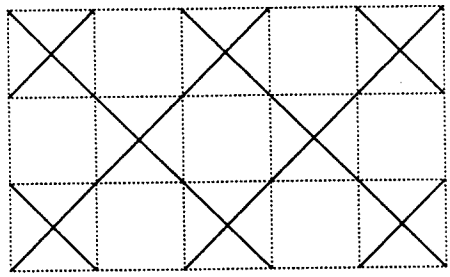

Fig. 5. Particle trajectories in the saturated, alternating, rescaled model.

the $\mathbf{Z}_N$ lattice the state of the system at time t is $\sigma(t) := (\sigma_1(t),..., \sigma_N(t))$, where $\sigma_{N+1} \equiv \sigma_1$ and N is even, since the model is alternating. There are, clearly, 2^N possible states.

Let $\langle \sigma(i+1) \mid \sigma(i) \rangle$ denote the probability that a particular state $\sigma(i)$ evolves to another particular state $\sigma(i+1)$. (The notation is designed to recall the bra–ket notation of quantum mechanics as well as the notation for conditional probability.) This state-to-state transition probability is just the product of the local transition probabilities: when $\sigma_1(i)$ is right-moving, $\langle \sigma(i+1) \mid \sigma(i) \rangle$ is given by

$$\prod_{n=1}^{N/2} R_{YZWX} \delta(W, \sigma_{2n-1}(i)) \, \delta(X, \sigma_{2n}(i)) \times \delta(Y, \sigma_{2n-1}(i+1)) \, \delta(Z, \sigma_{2n}(i+1))$$

and when $\sigma_1(i)$ is left-moving, it is

$$\prod_{n=1}^{N/2} R_{YZWX} \delta(W, \sigma_{2n}(i)) \, \delta(X, \sigma_{2n+1}(i)) \times \delta(Y, \sigma_{2n}(i+1)) \, \delta(Z, \sigma_{2n+1}(i+1))$$

It is convenient to regard each of these expressions as the components of $2^N \times 2^N$ matrices U^+ and U^-, global evolution matrices analogous to the local transition matrix R.[12,18,19]

More compactly, let W_n denote a two-dimensional vector space at node n with basis labeled by A and $\bar{A}$; the evolution matrices then act on the 2^N-dimensional vector space $W_1 \otimes \cdots \otimes W_N$. If we define R_{mn} to act on this space as the identity except on W_m and W_n, where it acts as the scattering operator R (in components

$$(R_{mn})_{\sigma'\sigma} := R_{\sigma'_m \sigma'_n \sigma_m \sigma_n} \prod_{j \neq m,n} \delta_{\sigma'_j \sigma_j}$$

where the subscript $\sigma'\sigma$ denotes that component of the $2^N \times 2^N$ matrix R_{mn}), then

$$U^+ := R_{12} R_{34} \cdots R_{N-1,N}$$
$$U^- := R_{23} R_{45} \cdots R_{N,1}$$

Now, to compute the probability of the system evolving from $\sigma(0)$ to $\sigma(t)$, we must add the probability of each possible evolution path

$$\langle \sigma(t) \mid \sigma(t-1) \rangle \langle \sigma(t-1) \mid \sigma(t-2) \rangle \cdots \langle \sigma(1) \mid \sigma(0) \rangle$$

That is,

$$\langle \sigma(t) \mid \sigma(0) \rangle = \sum_{\sigma(1)} \cdots \sum_{\sigma(t-1)} \prod_{i=0}^{t-1} \langle \sigma(i+1) \mid \sigma(i) \rangle \tag{1}$$

This should be seen as analogous to a path integral in quantum field theory, although here our system is discrete and we are computing real probabilities rather than complex quantum amplitudes. Writing

$$\langle \sigma(i+1) \mid \sigma(i) \rangle =: \exp[-E(\sigma(i+1), \sigma(i))/kT]$$

for some (formal) scalar kT makes the analogy clearer, for then

$$\langle \sigma(t) \mid \sigma(0) \rangle = \sum_{\sigma(1)} \cdots \sum_{\sigma(t-1)} \exp\left[-\frac{1}{kT} \sum_{i=0}^{t-1} E(\sigma(i+1), \sigma(i)) \right] \tag{2}$$

which is essentially the sum over paths/histories of the action of a path/history.

Since $\langle \sigma(i+1) \mid \sigma(i) \rangle$ is also a product, this suggests rewriting its factor R as an exponential, too. Writing R_{YZWX} as $\exp[-E(W, X, Y, Z)/kT]$ means that $E(\sigma(i+1), \sigma(i))$ is the sum

$$\sum_{n=1}^{N/2} E(\sigma_{2n-1}(i), \sigma_{2n}(i), \sigma_{2n-1}(i+1), \sigma_{2n}(i+1))$$

or

$$\sum_{n=1}^{N/2} E(\sigma_{2n}(i), \sigma_{2n+1}(i), \sigma_{2n}(i+1), \sigma_{2n+1}(i+1))$$

according to whether $\sigma_1(i)$ is right- or left-moving. Inserting these expressions into the exponent in (2), we find that it becomes

$$-\frac{1}{kT} \sum_{\text{interactions}} E(\text{interaction})$$

where the interactions are those shown in Fig. 5. Thus, (2) is exactly the *partition function* for a statistical mechanics model on the particle trajectory lattice with states being assignments of A or $\bar{A}$ to each edge, Boltzmann weights at each vertex defined by $\exp[-E(\text{interaction})/kT]$ and boundary conditions of $\sigma(0)$ and $\sigma(t)$.

6. SOLVING THE MODEL

Having made this identification between the lattice gas automaton and this statistical mechanics model, we recognize U^+ and U^- as the diagonal-to-diagonal transfer matrices used in the exact solution of such statistical mechanics models.[12] The sums and products in the expression in (1) for the path integral/partition function are no more than the product of these transfer matrices:

$$\langle \sigma(t) \mid \sigma(0) \rangle = (U^- U^+ \cdots U^- U^+)_{\sigma(t)\,\sigma(0)}$$

(assuming the initial state has a right-moving particle at node 1), where the subscript $\sigma(t)\,\sigma(0)$ again denotes that component of the $2^N \times 2^N$ matrix obtained by taking the product of the U's. It would be easy to evaluate this product analytically if the matrices could be diagonalized. Failing this, the less specific problem of evaluating the trace of the matrix on the right-hand side of the equation (i.e., evaluating the partition function for the statistical mechanics model on a torus) in the limit as $t \to \infty$ requires only the knowledge of the largest eigenvalue of $U := U^- U^+$. Since commuting matrices have the same eigenvalues and a common set of eigenvectors, one is motivated to search for matrices which commute with the transfer matrices.

Following Destri and de Vega,[18] begin by defining the one-step translation operator on the spatial lattice,

$$V := P_{12} P_{34} \cdots P_{N-1,N}$$

where P_{mn} is the permutation operator, which is given by R_{mn} when $p = 1$ and $q = 0$. Just as do $U^\pm$, V acts on $W_1 \otimes \cdots \otimes W_N$; in fact, the evolution operators differ by conjugation with V:

$$U^\pm = V U^\mp V^{-1}$$

which implies that V^2 commutes with the evolution:

$$[U^\pm, V^2] = 0 = [U, V^2]$$

This is not surprising, since a shift by an even number of lattice points is a symmetry of the model.

What other operators on $W_1 \otimes \cdots \otimes W_N$ might commute with $U^\pm$? The simplest possibility is a tensor product of N operators, each acting on one of the W_n. Less trivially, consider 2×2 matrices of operators, $L_n^\pm$, which act at node n when it carries a right (left)-moving particle. Then the matrix products

$$\tau^\pm := L_1^\pm L_2^\mp \cdots L_{N-1}^\pm L_N^\mp$$

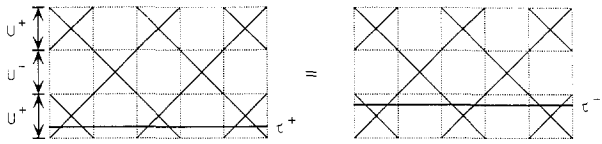

Fig. 6. The monodromy operators τ^+ and τ^- act before and after the row of interactions U^+, respectively.

known in the quantum inverse scattering method as *monodromy* matrices, are 2×2 matrices of operators on $W_1 \otimes \cdots \otimes W_N$. Since

$$\tau^{\pm} = V \tau^{\mp} V^{-1}$$

the traces of the monodromy matrices are the same operator, T (the row-to-row transfer matrix), acting on $W_1 \otimes \cdots \otimes W_N$. If we take $L_n^{\pm}$ to act on $W_0 \otimes W_n$, where W_0 is an auxiliary two-dimensional vector space, $\tau^{\pm}$ and $\tau^{\mp}$ act before and after a row of interactions, namely $U^{\pm}$, as shown in Fig. 6. The result is the same, i.e.,

$$U^{\pm} \tau^{\pm}_{\sigma_0 \sigma_0'} = \tau^{\mp}_{\sigma_0 \sigma_0'} U^{\pm}$$

which implies $[U^{\pm}, T] = 0$, provided the action of a pair of L's commutes with that of a single R, as shown in Fig. 7. This is known as the star–triangle or Yang–Baxter condition:

$$L_n^+ L_{n+1}^- R_{n,n+1} = R_{n,n+1} L_n^- L_{n+1}^+ \tag{3}$$

Should there exist a nontrivial class of solutions to this equation, there whould be the desired collection of transfer matrices T commuting with $U^{\pm}$.

At this point we note that the scattering matrix R is a special case of the Boltzmann weight of the zero-field symmetric eight-vertex model:

$$W := \begin{array}{c} \\ AA \\ A\bar{A} \\ \bar{A}A \\ \bar{A}\bar{A} \end{array} \begin{array}{c} \begin{array}{cccc} AA & A\bar{A} & \bar{A}A & \bar{A}\bar{A} \end{array} \\ \begin{pmatrix} a & 0 & 0 & d \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ d & 0 & 0 & a \end{pmatrix} \end{array}$$

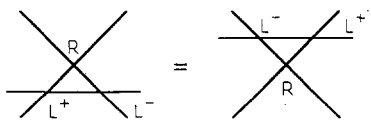

Fig. 7. Graphical representation of the star–triangle or Yang–Baxter equation.

This more general model has been solved by Baxter[12]: the result is that the Yang–Baxter equation is satisfied by any three operators having the same form as W with equal values of the two parameters

$$\Delta := \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

$$\Gamma := \frac{ab - cd}{ab + cd}$$

There is thus one degree of freedom in the solution to (3) beyond a trivial normalization factor, and hence there is a one-parameter family of mutually commuting transfer matrices as desired. This means our model is solvable; its computable properties and their implications will be discussed in the final section.

7. DISCUSSION

For a symmetric eight-vertex model with Boltzmann weight R, the parameter Δ is given by

$$\Delta = \frac{p^2 + (1-q)^2 - q^2 - (1-p)^2}{2[p(1-q) + q(1-p)]} = \frac{p - q}{p + q - 2pq}$$

It is easy to check that $|\Delta| \leqslant 1$ for $0 \leqslant p, q \leqslant 1$. Now, when $|\Delta| < 1$ the symmetric eight-vertex model is in its disordered regime: one can show that there is no spontaneous magnetization or polarization. Further, when $a + d = b + c$, as in this case where the parameters are probabilities, the correlation length vanishes: the system is at a point of complete disorder. Specialization to the case when $d = 0$ defines the six-vertex model. Simple transformations from any of the cases p or q equals one or zero (as with a deterministic rule) place us in this situation. In each of these cases $\Delta = 1$ and the model lies on a first-order critical line between disordered and totally ordered regimes of the six-vertex model. This is a familiar result for probabilistic cellular automata,[10,20,21] although we have arrived at it by a slightly different route than is usually followed; the correlation function for this situation has been calculated in ref. 22.

There are several observations to be made. These results apply to ensembles of systems, in the thermodynamic limit. That the correlation length vanishes, for example, does not mean that there is no correlation between states of the probabilistic lattice gas automaton at successive time steps. Nor would a nonvanishing correlation length mean that there were necessarily spacelike correlations in a given initial state. Instead, the inter-

pretation is that deep in the interior of a system, far from the boundary conditions and thus/or subject to essentially random boundary conditions, a domain of sufficiently large size is in thermodynamic equilibrium. Of course, unless the domain is infinite, i.e., the thermodynamic limit results apply strictly, finite-size effects will modify the results—the correlation length will not vanish exactly, say, but only be very small.[23] Since these results are obtained for specific regimes of the full symmetric eight-vertex model, they can be thought of as shadows of quantum mechanical behavior in the probabilistic or even deterministic model, which are visible in the ensemble limit. A major open question is to understand the relation between the quantum Hamiltonian which can be constructed from the transfer matrix in these models[12,19,24] and the classical equations of motion which are supposed to be approximated in the macroscopic limit; our results only provide evidence for the validity of the first step in the derivation of the latter—the assumption of local thermodynamic equilibrium. While it seems clear that two-dimensional thermodynamic equilibrium should be impossible without thermodynamic equilibrium in any one-dimensional subspace, this also remains to be proved.

Note that although thermodynamic equilibrium obtains, the original model is explicitly nonergodic—recall the split into two uncoupled (even and odd) systems. Our success in analyzing the model this way suggests that this approach may be appropriate for other lattice gas models: Rather than trying to break spurious conservation laws in a discrete dynamical system, provided that the conserved quantity does not have a continuum limit, perhaps one should use it to partition the model and then concentrate on the components separately.[17] In general, this model demonstrates again the efficacy of constructing and analyzing lattice gas automata according to (spacetime) symmetry principles; we expect that this approach should prove complementary to other approaches, such as Markov processes,[15] for obtaining thermodynamic results for more realistic lattice gas models for fluid flow.

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